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MRC Technical Summary Report #2158

A NEW PROOF AND AN EXTENSION OF A THEOREM OF P. RABINOWITZ CONCERNING NONLINEAR WAVE EQUATIONS

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610 Walnut Street
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December 1980

Received November 17, 1980

Approved for public release
Distribution unlimited

Sponsored by

U. S. Army Research Office
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81 3 19 073

UNIVERSITY OF WISCONSIN - MADISON
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ABSTRACT

In 1978, P. Rabinowitz proved a theorem concerned with the existence of a nontrivial periodic solution of a nonlinear wave equation with a continuous increasing superlinear nonlinear term. In this paper we present a new and simpler proof of this theorem and relax an assumption on the nonlinear term, which is a discontinuous nondecreasing function.

AMS(MOS) Subject Classification: 35L05, 35L60, 35B10, 49G99.

Key words: nonlinear wave equation, mountain pass lemma, critical point, duality argument, generalized gradient.

Work Unit No. 1 - Applied Analysis.

†Partially supported by the University of Wisconsin Graduate School Research Committee.

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

SIGNIFICANCE AND EXPLANATION

The existence of a nontrivial periodic solution of a nonlinear wave equation was obtained by P. Rabinowitz. We present a new and simpler proof and extend the result to a more general case.

The responsibility for the wording and views expressed in this descriptive summary lies with MRC, and not with the authors of this report.

A NEW PROOF AND AN EXTENSION OF A THEOREM OF P. RABINOWITZ CONCERNING NONLINEAR WAVE EQUATIONS.

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This paper concerns a new simpler proof of the existence of a nontrivial periodic solution of a nonlinear wave equation. Namely, we prove the following theorem:

<u>Theorem</u> Suppose that

(G_1) $g: R^1 \to R^1$ is a nondecreasing function such that $g^{-1}(0)$ is a closed interval including 0 .

(G_2) $\exists\ \theta \in [0, 1/2)$ and a constant C_0 such that

$$G(t) := \int_0^t g(s)ds \le \begin{cases} \theta\, t\, g(t-0) & \text{for } t > C_0 > 0 \\ \theta\, t\, g(t+0) & \text{for } t < -C_0 < 0 \end{cases}$$

(G_3) $$\lim_{t\to 0} g(t)/t = 0 .$$

Then there exists a nontrivial (weak) solution $u \in L^\infty$ of

$$\text{(I)}\quad \begin{aligned} &u_{tt} - u_{xx} + g(u) = 0 \quad \text{for } (x,t) \in \Omega := (0,\pi) \times (0,2\pi) && (1) \\ &u(0,t) = u(\pi,t) = 0 && (2) \\ &u(x,0) = u(x,2\pi) && (3) \end{aligned}$$

It is well known, that the pioneering work of P. Rabinowitz [1] made a breakthrough and stimulated great interest in this problem. The assumption (G_1) in our theorem is slightly weaker than that in [1], where g is assumed to be continuous. Recently, H. Brézis, J. M. Coron and L. Nirenberg [2]

†Partially supported by the University of Wisconsin Graduate School Research Committee.

Sponsored by the United States Army under Contract No. DAAG29-80-C-0041.

presented a simpler proof of the theorem of Rabinowitz, under slightly different conditions. Their method of proof, which is based on a duality argument and a modified (P. S. condition and) mountain pass lemma, is quite powerful. After learning of their paper, we tried to better understand their method, and to modify their proof so that one does not have to change the (P. S. condition and the) mountain pass lemma. There are two main differences between the techniques of [2] and this paper: (1) In [2], the inverse function of g is truncated into a bounded function, but in this paper, it is truncated into a power function, as a result of which the mountain pass condition becomes easier to check and it is not necessary to assume the period is small; (2) An ε-perturbation technique is used here, in order to ensure that the P. S. condition holds. It seems that our technique is easier to extend to case where $g(u)$ is discontinuous.

Let $Au = u_{tt} - u_{xx}$ be the wave operator acting on functions in L^1 satisfying (2), (3). Let $N(A)$ be the kernel of A, and $R(A)$ be the range of A. It is known [2] that

$$N(A) = \{p(t+x) - p(t-x) \mid p \text{ is } 2\pi\text{-periodic, } p \in L^1_{loc}(R) \text{ and } \int_0^{2\pi} p = 0\}.$$

Let K be the inverse of A, K is defined on $R(A)$, and bounded in the following sense [2]:

$$\|Ku\|_{L^\infty} \leq C_1 \|u\|_{L^1}. \tag{4}$$

Let $\hat{g}$ be the maximal monotone graph associated with the nondecreasing function g, i.e. $\hat{g}$ is the set-valued map: $u \to [g(u-0), g(u+0)]$. Let $\hat{h}$ be the inverse maximal monotone graph of $\hat{g}$, i.e. for $u \in \hat{h}(v)$ if and only if $v \in \hat{g}(u)$. According to the assumption (G_2), $g(u) \to \pm\infty$ as $u \to \pm\infty$ (see below (23)), except if $g \equiv 0$ (but in this case the existence of nontrivial solutions is obvious), therefore to define a nondecreasing function $h : R^1 \to R^1$ such that $\hat{h}(v) = [h(v-0), h(v+0)]$.

According to the assumption (G_1), $0 \varepsilon \hat{h}(0)$.

1° <u>As in</u> [2] , <u>we begin by converting our problem (I) into the following problem</u>:

(II) <u>Find</u> $v \varepsilon E := \{v \varepsilon L^{p'}(\Omega) | \int v\varphi = 0 \quad \forall \varphi \varepsilon L^p(\Omega) \cap N(A))\}$ <u>where</u> $\frac{1}{p} + \frac{1}{p'} = 1$, $p \varepsilon (2, \frac{1}{\theta})$ and $\chi \varepsilon N(A)$ <u>such that</u> v, χ satisfy

$$\chi \varepsilon K v + \hat{h}(v)$$

For if v is a nontrivial solution of (5) , set $u = \chi - Kv$, we have

$$\begin{cases} Au + v = \hat{} \\ u \varepsilon \hat{h}(v) . \end{cases}$$

Then

$$Au + \hat{g}(u) \ni 0 .$$

Then by Theorem 3.1 in [3] , u is also a nontrivial solution of the problem (I).

2°. Before solving the problem (II) we consider a <u>truncated and perturbed problem:</u>

Find $v \varepsilon E$, $\chi \varepsilon N(A)$ such that

$$\chi \varepsilon Kv + \hat{h}_{M,\varepsilon}(v) \tag{6}$$

where

$$h_{M,\varepsilon}(v) = h_M(v) + \varepsilon |v|^{p'-2} v$$

and

$$h_M(v) = \begin{cases} h(M)[1 + \frac{1}{M^{p'-1}} (v-M)^{p'-1}] & \text{if} \quad v > M \\ h(v) & \text{if} \quad |v| \leq M \\ h(-M)[1 + \frac{1}{M^{p'-1}} |v+M|^{p'-1}] & \text{if} \quad v < -M . \end{cases}$$

Let

$$H_{M,\varepsilon}(v) = \int_0^v h_{M,\varepsilon}(s)ds . \tag{7}$$

$H_{M,\varepsilon}(v)$ is a convex function, provided that $h_{M,\varepsilon}$ is nondecreasing. We look for critical points of the following functional on E

$$f_{M,\varepsilon}(v) = \tfrac{1}{2} \int_\Omega K\, v \cdot v + \int_\Omega H_{M,\varepsilon}(v). \tag{8}$$

It is worth pointing out that the functional $f_{M,\varepsilon}$ is not differentiable on the space E , but only locally Lipschitzian. The critical point v_0 of $f_{M,\varepsilon}$ is understood in the generalized sense, i.e., $\theta \in \partial f_{M,\varepsilon}(v_0)$ where ∂f is the generalized gradient of f (cf. [4][5]).

Note that

$$\partial f_{M,\varepsilon}(v) \subset \partial\, \tfrac{1}{2} \int_\Omega Kv \cdot v + \partial \int_\Omega H_{M,\varepsilon}(v)$$

$$= Kv + \partial \int_\Omega H_{M,\varepsilon}(v)$$

The functional $v \to \int_\Omega H_{M,\varepsilon}(v)$ is convex and the generalized gradient for a convex functional coincides with its subdifferential. The Hahn Banach separation theorem implies that the subdifferential of $\int H_{M,\varepsilon}(v)$ on the space E is equal to $\hat{h}_{M,\varepsilon}(v) - \chi$ where $\chi \in N(A)$, This proves

$$\partial f_{M,\varepsilon}(v) \subset Kv + \hat{h}_{M,\varepsilon}(v) - \chi .$$

Therefore, if v_0 is a critical point of $f_{M,\varepsilon}$, then it solves the relation (6).

3°. Now we apply the mountain pass lemma (for locally Lip. functionals (cf. [4], Theorem 3.4) to prove the existence of a nontrivial critical point of $f_{M,\varepsilon}$.

The mountain pass lemma reads as follows:

Let X be a Banach space, let f be a locally Lipschitzian function defined on X . Assume the following conditions on f are satisfied:

(1). The $(P.A.)^+$ condition. i.e. Any sequence $\{x_n\}$ in X , along which

$\exists$ constants α_1, α_2 such that $0 < \alpha_1 \leq f(x_n) \leq \alpha_2$ and

$$\lambda(x_n) := \underset{w_n \in \partial f(x_n)}{\mathrm{Min}} \|w_n\| \to 0 \quad \text{as} \quad n \to \infty$$

possesses a convergent subsequence.

(2) There exists $r, \rho > 0$ with $f > 0$ in the ball without $\theta : B_r \setminus \theta$ and $f|_{S_r} \geq \rho$ where S_r is the sphere with radius r.

(3) There is an $e \in X$, $e \neq \theta$ such that $f(e) = 0$. Then f has a nontrivial critical point.

<u>Verification of the $(P.\,S.)^+$ condition.</u>

Let $\{v_n\}$ be a sequence in E such that

$$0 < \alpha_1 \leq f_{M,\varepsilon}(v_n) \leq \alpha_2 \tag{9}$$

$$\lambda(v_n) := \underset{w_n \in \partial f_{M,\varepsilon}(v_n)}{\mathrm{Min}} \|w_n\|_{L^p} \to 0 \quad \text{as} \quad n \to \infty. \tag{10}$$

We shall prove that v_n possesses a convergent subsequence in E.

(1) Firstly, we show v_n is bounded : $\|v_n\|_{L^{p'}} \leq C$ (in the following, various constants are all denoted by C if there is no confusion). We have

$$\alpha_1 \leq \int_\Omega \tfrac{1}{2}(Kv_n)v_n + \int_\Omega H_{M,\varepsilon}(v_n) \leq \alpha_2 \tag{11}$$

$$w_n + \chi_n \in K\, v_n + \hat{h}_{M,\varepsilon}(v_n) \tag{12}$$

where $\|w_n\|_{L^p} = \lambda(v_n)$ and $\chi_n \in N(A)$. Equation (12) implies that

$\exists\ \xi_n = \eta_n + \varepsilon|v_n|^{p'-2}v_n$ with $\eta_n \in \hat{h}_M(v_n)$ such that

$$w_n + \chi_n = K\, v_n + \xi_n\,. \tag{13}$$

Combining (11) with (13), we obtain

$$\int H_{M,\varepsilon}(v_n) - \tfrac{1}{2}\xi_n \cdot v_n \leq \alpha_2 + \tfrac{1}{2}\lambda(v_n)\|v_n\|_{L^{p'}} . \qquad (14)$$

However, there is a constant C_M, such that

$$(\sigma_M^{\pm} + \varepsilon)|v|^{p'-2}v - C_M \leq h_{M,\varepsilon}(v-0) \leq h_{M,\varepsilon}(v+0) \leq (\sigma_M^{\pm} + \varepsilon)|v|^{p'-2}v + C_M \quad (15)$$

for $v >$ (or $<$) 0, where $\sigma_M^{\pm} = \frac{\pm 1}{M^{p'-1}} h(\pm M)$, and hence

$$H_{M,\varepsilon}(v) \geq (\sigma_M + \varepsilon)\frac{|v|^{p'}}{p'} - C_M|v| \qquad (16)$$

where $\sigma_M = \min\{\sigma_M^+, \sigma_M^-\}$. Substituting (15), (16) into (14), we have

$$\left(\frac{1}{p'} - \tfrac{1}{2}\right)(\sigma_M + \varepsilon)\int_\Omega |v_n|^{p'} - \frac{3}{2}\int_\Omega C_M|v_n| \leq \alpha_2 + \tfrac{1}{2}\lambda(v_n)\|v_n\|_{L^{p'}} .$$

Thus,

$$\|v_n\|_{L^{p'}} \leq C . \qquad (17)$$

(2) Secondly, since v_n possesses a weakly convergent subsequence, which we still denote by v_n, suppose $v_n \rightharpoonup v^*$, the following relation:

$$\lim_{n,m\to\infty} \int_\Omega \xi_n \cdot v_m = -\int_\Omega Kv^* \cdot v^* \qquad (18)$$

follows from (13) and the compactness of K.

(3) Finally, we prove the subsequence v_n converges to v^* strongly in $L^{p'}$. If not, $\delta > 0$ and subsequences $\{v_{n_i}\}$, $\{v_{m_i}\}$ such that

$$\|v_{n_i} - v_{m_i}\|_{L^{p'}} \geq \delta \quad \forall\, i .$$

On the one hand, (18) implies

$$\int(\xi_{n_i} - \xi_{m_i})(v_{n_i} - v_{m_i}) \to 0 \quad \text{as} \quad i \to \infty . \tag{19}$$

On the other hand, since $h_M(v)$ is nondecreasing,

$$\int_\Omega (\xi_{n_i} - \xi_{m_i})(v_{n_i} - v_{m_i}) = \int_{\{y\in\Omega | v_{n_i}(y) > v_{m_i}(y)\}} + \int_{\{y\in\Omega | v_{n_i}(y) < v_{m_i}(y)\}}$$

$$\geq \varepsilon \int_\Omega (|v_{n_i}|^{p'-2} v_{n_i} - |v_{m_i}|^{p'-2} v_{m_i})(v_{n_i} - v_{m_i})$$

$$= \varepsilon \int_\Omega |\lambda v_{n_i}(y) + (1-\lambda)v_{m_i}(y)|^{p'-2}(v_{n_i}(y) - v_{m_i}(y))^2$$

$$\geq \frac{\varepsilon \left(\int_\Omega |v_{n_i}(y) - v_{m_i}(y)|^{p'}\right)^{\frac{2}{p'}}}{\left(\int_\Omega |\lambda v_{n_i}(y) + (1-\lambda)v_{m_i}(y)|^{p'}\right)^{\frac{2-p'}{p'}}} \geq \frac{\varepsilon \delta^2}{C^{2-p'}} > 0 \text{ (Hölder inequality)}$$

for some $\lambda \in (0,1)$. This contradicts (19). Thus $v_n \to v^*$ in $L^{p'}$ strongly.

<u>Verification of the condition:</u>

$$f_{M,\varepsilon}(v)|_{S_r} \geq \rho > 0 , \ \underline{\text{for small}} \ r > 0 ,$$

where ρ is a constant, independent of ε, and S_r is the sphere with radius r.

According to the assumption (G_3), we have

$$\lim_{v\to 0} \frac{H_M(v)}{v^2} = +\infty . \tag{21}$$

Combining (21) with (16) yields

$$H_M(v) \geq \begin{cases} c_\delta \, v^2 & \text{if} \quad |v| \leq \delta \\ \delta c_\delta |v| & \text{if} \quad \delta < |v| < T \\ \frac{\sigma_M}{2} |v|^{p'} & \text{if} \quad |v| \geq T \end{cases}$$

where $C_\delta \to +\infty$ as $\delta \to 0$, and $T = \left(\frac{C_M}{(\frac{1}{p'} - \frac{1}{2})\sigma_M}\right)^{\frac{1}{p'-1}}$. By (4)

$$f_{M,\varepsilon}(v) \geq -\frac{C_1}{2} \|v\|^2_{L^1} + C_\delta \int_{|v|\leq\delta} v^2 + \delta \, C_\delta \int_{\delta<|v|<T} |v| + \frac{\sigma_M}{2} \int_{|v|\geq T} |v|^{p'} .$$

Since

$$\Big(\int_{|v|\leq\delta} |v| \Big)^2 \leq \text{mes}(\Omega) \int_{|v|\leq\delta} |v|^2 ,$$

it follows that

$$\int_{|v|\geq T} |v| \quad \leq \text{mes}(\Omega)^{\frac{1}{p}} \Big(\int_{|v|\geq T} |v|^{p'} \Big)^{\frac{1}{p'}} ,$$

$$\int_{|v|\leq\delta} |v|^{p'} \leq \Big(\int_{|v|\leq\delta} v^2 \Big)^{\frac{p'}{2}} \text{mes}(\Omega)^{1-p'/2} ,$$

$$\frac{1}{T^{p'-1}} \int_{\delta\leq|v|\leq T} |v|^{p'} \quad \leq \int_{\delta\leq|v|\leq T} |v| ,$$

and that $\int_\Omega |v| < 1$ for $\Big(\int_\Omega |v|^{p'}\Big)^{\frac{1}{p'}} = r$ with r sufficiently small.

Firstly, we fix $\delta > 0$ such that $C_\delta > \frac{3}{2} C_1 \text{ mes}(\Omega) + \text{mes}(\Omega)^{\frac{2}{p'}-1}$ and then choose r small enough such that

$$\sigma C_\delta \int_{\delta<|v|<T} |v| - \frac{3}{2} C_1 \Big(\int_{\delta<|v|<T} |v| \Big)^2 \geq \Big(\int_{\delta<|v|<T} |v|^{p'} \Big)^{\frac{2}{p'}}$$

$$\frac{\sigma_M}{2} \int_{|v|>T} |v|^{p'} - \frac{3}{2} C_1 \Big(\int_{|v|>T} |v| \Big)^2 \geq \Big(\int_{|v|>T} |v|^{p'} \Big)^{\frac{2}{p'}} .$$

It follows that

$$f_{M,\varepsilon}(v) \geq \Big(\int_{|v|<\delta} |v|^{p'} \Big)^{\frac{2}{p'}} + \Big(\int_{\delta<|v|<T} |v|^{p'} \Big)^{\frac{2}{p'}} + \Big(\int_{|v|>T} |v|^{p'} \Big)^{\frac{2}{p'}}$$

$$\geq c_p \Big(\int_\Omega |v|^{p'} \Big)^{\frac{2}{p'}} = c_p r^2 \qquad \text{for } \|v\|_{L^{p'}} = r$$

where c_p is a constant depends on p. Let $\rho = c_p r^2$, which is a constant independent of ε.

<u>Verification of the condition</u>

$\exists\, e \in E$ <u>such that</u> $f_{M,\varepsilon}(e) = 0$ <u>with</u> $e > r$.

In fact, taking $e_t = t\varphi_1$, where φ_1 is an eigenfunction of K, associated with a negative eigenvalue, let $\lambda_1 = -\int K \varphi_1 \varphi_1 > 0$. Therefore

$$f_{M,\varepsilon}(e_t) \leq -\frac{\lambda_1}{2} t^2 + \frac{t^{p'}}{p'} (\sigma'_M + \varepsilon) \int |\varphi_1|^{p'} + C_M t \int |\varphi_1| \to -\infty, \quad \text{as } t \to \infty ,$$

where $\sigma'_M = \frac{1}{M^{p'-1}} \max\{h(M) - h(-M)\}$. Thus $\exists$ a $t > 0$ such that

$$f_{M,\varepsilon}(t\varphi_1) = 0 , \quad \text{with } \|t\varphi_1\| > r .$$

Applying the mountain pass lemma, there exists a critical point $v_{M\varepsilon}$ of $f_{M\varepsilon}$ such that

$$f_{M,\varepsilon}(v_{M,\varepsilon}) \geq \rho .$$

4° Return to the unmodified problem

By the characterization of the critical value

$$f_{M,\varepsilon}(v_{M,\varepsilon}) \leq \max_{s\varepsilon[0,1]} f_{M,\varepsilon}(se) \leq C \ , \tag{22}$$

where constant C does not depend on ε or M. In fact, with no loss of generality, we assume $0 < \theta < 1/2$ and $C_0 > 1$ in the assumption (G_2), i.e., $G(u) \leq \theta\, u\, g(u-0)$ as $t > C_0$.

By an approximation procedure using differentiable functions, we see

$$C_1\, u^{\frac{1}{\theta}} \leq G(u) \quad \text{where } C_1 \text{ is a constant.}$$

Thus

$$g(u-0) \geq \frac{C_1}{\theta}\, u^{\frac{1}{\theta}-1} \geq \frac{C_1}{\theta}\, u^{p-1} \tag{23}$$

and we have

$$h(v+0) \leq C_2\, v^{p'-1} \tag{24}$$

where $C_2 = \left(\frac{\theta}{C_1}\right)^{p'-1}$. This implies

$$H(v) \leq C_2\, v^{p'} + C_3 \qquad \forall\ v > 0 \ .$$

Similarly, for $v < 0$, we have

$$h(v-0) \geq -C_2 |v|^{p'-1} \tag{24)'$$

and

$$H(v) \leq C_2 |v|^{p'-1} + C_3 \ .$$

Since

$$H_M(v) = \begin{cases} H(v) & \text{if } |v| \leq M \\ H(v) \pm h(\pm M)\left[\, |v \pm M| + \frac{1}{p' M^{p'-1}} |v \pm M|^{p'} \right] & \text{if } |v| > M \end{cases}$$

we have

$$H_M(v) \leq C_4 |v|^{p'} + C_5 \tag{25}$$

where C_4 and C_5 are constants, independent of M. Substituting (25) into (23), we see

$$f_{M,\varepsilon}(v_{M,\varepsilon}) \leq \operatorname*{Max}_{s\varepsilon[0,1]} f_{M,\varepsilon}(se) \leq \operatorname*{Max}_{t\varepsilon[0,\infty]} f_{M,\varepsilon}(t\varphi_1)$$

$$\leq \operatorname*{Max}_{t\varepsilon[0,\infty)} [-\frac{\lambda_1}{2} t^2 + C_6(C_4+1)t^{p'} + C_7] \leq C_8 .$$

The constant C_8 is independent of ε and M i.e. (22) holds.

Repeating the procedure performed in verifying the P.S. condition, we obtain

$$\|v_{M,\varepsilon}\|_{L^{p'}} \leq C \quad (C \text{ does not depend on } \varepsilon \text{ and } M \text{ too}).$$

Therefore there is a weakly convergent subsequence $v_{\varepsilon_i} \to v_M$, with $\|v_M\|_{L^{p'}} \leq C$. We are going to prove that v_M is a nontrivial critical point of $f_{M,0}$. In fact, by the monotonicity of h_M, we have

$$\int (\xi_i - \xi)(v_{\varepsilon_i} - \zeta) \geq 0 \qquad \forall \zeta \in E$$

for each $\xi_i \in \hat{h}_M(v_{\varepsilon_i})$, $\xi \in \hat{h}_M(\zeta)$. Since

$$\exists \chi \in N(A) \text{ such that } \chi \in K v_{M,\varepsilon} + \hat{h}_{M,\varepsilon}(v_{M,\varepsilon})$$

we have

$$\int(-Kv_{\varepsilon_i} - \varepsilon_i|v_{\varepsilon_i}|^{p'-2}v_{\varepsilon_i} - \xi)(v_{\varepsilon_i} - \zeta) \geq 0 .$$

Set $\zeta = \zeta_M = v_M + t\eta$, $\forall \eta \in E$. Due to the weakly upper semi-continuity of the set-valued map $\zeta_M \mapsto \hat{h}_M(\zeta_M)$ (cf. [4] §1, prop. (6)), we get $\xi_M \in \hat{h}_M(v_M)$ such that

$$\int (K v_M + \xi_M)\eta \geq 0 \qquad \forall \eta \in E$$

i.e. $\exists \chi \in N(A)$ such that

$$\chi \in K v_M + \hat{h}_M(v_M).$$

According to the convexity of H_M and the lower semi-continuity of $\int H_M(v)$, it follows

$$f_{M,0}(v_M) = \lim_{i\to\infty} f_{M,\varepsilon_i}(v_{\varepsilon_i}) \geq \rho > 0 .$$

Lastly in order to get rid of the influence of M, we shall prove the following

5° <u>a priori estimate:</u> $\exists$ <u>constant C, independent of M such that</u>

$$\|v_M\|_{L^\infty} \leq C . \qquad (26)$$

If (26) holds, then v_M is a solution of the problem II for $M > C$.

It has already been proved that

$$\|v_M\|_{L^{p'}} \leq C ; \qquad (27)$$

and by (24) and (24')

$$|h(v \pm 0)| \leq C_2 |v|^{p'-1} .$$

Thus

$$|h_M(v \pm 0)| \leq C_3 |v|^{p'-1}$$

which implies

$$\|h_M(v_M \pm 0)\|_{L^p} \leq C_4 . \qquad (28)$$

Moreover, it follows from (4) and (27) that

$$\|K v_M\|_{L^\infty} \leq C_5 \qquad (29)$$

so we obtain

$$\|\chi_M\|_{L^p} \leq C_6 \quad \text{for} \quad \chi_M \in K v_M + \hat{h}_M(v_M) . \qquad (30)$$

By the definition of h_M, in order to get the estimate (26), we only need an L^∞ a priori bound for χ_M, i.e.

$$\|\chi_M\|_{L^\infty} \leq C_7 \qquad (31)$$

According to [6]

$$\chi_M(x,t) = q_M(x + t) - q_M(t - x) \tag{32}$$

where

$$q_M(t) = \frac{1}{2\pi} \int_0^\pi [\chi_M(x,t-x) - \chi_M(x,t+x)]\,dx \ . \tag{33}$$

We know $q_M \in L^p(0,2\pi)$ from (30), so that

$$\|q_M\|_{L^1(0,2\pi)} < C_8 \ .$$

It is easily seen that

$$\int_0^\pi [v_M(x,t-x) - v_M(x,t+x)]\,dx = 0 \quad \text{for a.e.t} \tag{34}$$

and from (29) and (32) we have

$$-C_5 + q_M(t+x) - q_M(t-x) \le u_M(x,t) \le C_5 + q_M(t+x) - q_M(t-x) \ .$$

Let g_M be the inverse function of h_M, which coincides with g before truncation

$$g_M(-C_5 + q_M(t+x) - q_M(t-x)-0) \le v_M(x,t) \le g_M(C_5 + q_M(t+x) - q_M(t-x)+0).$$

One deduces from (34) that

$$\int_0^{2\pi} \tilde{g}_M(-C_5 + q_M(t) - q_M(s))\,ds \le 0 \qquad \text{a.e.t.} \tag{35}$$

where $\tilde{g}_M(u) = g_M(u-0) - g_M(-u+0)$ is an odd increasing function. Fixing t, such that $q_M(t) > 0$ we get

$$\tilde{g}_M(-C_5 + q_M(t) - q_M(s)) \ge \tilde{g}_M\left(\frac{q_M(t)}{2}\right) \quad \text{if} \quad q_M(s) + C_5 \le \frac{q_M(t)}{2}$$

$$\tilde{g}_M(-C_5 + q_M(t) - q_M(s)) \ge -\tilde{g}_M(q_M(s) + C_5) \quad \text{if} \quad q_M(s) + C_5 > \frac{q_M(t)}{2} \ .$$

Substituting these two inequalities into (35), one obtains

$$\tilde{g}_M\left(\frac{q_M(t)}{2}\right) \operatorname{mes}\{s \in (0,2\pi) \mid q_M(s) \le \frac{q_M(t)}{2} - C_5\} \le \int_0^{2\pi} \tilde{g}_M(q_M(s) + C_5)\,ds \quad \text{a.e.t} \tag{36}$$

Since $q_M \varepsilon L^{p-1}(0,2\pi)$ and $\tilde{g}_M$ is of the growth power $p-1$, the RHS of (36) is bounded by a constant C_M.

Firstly, we prove $q_M \varepsilon L^{\infty}(0,2\pi)$. In fact,

$\exists\, u_0$ such that $\tilde{g}_M(u_0) > \frac{C_M}{\pi}$

$\exists\, u_1$ such that $\operatorname{mes}\{s \varepsilon (0,2\pi) | q_M(s) < u_1\} > \pi$.

Let $n = \max\{2u_0, 2(u_1 + C_5)\}$. If $\operatorname*{ess\,sup}_{t \varepsilon (0,2\pi)} q_M(t)$ were not bounded, then the set $S = \{t \varepsilon (0,2\pi) | q_M(t) > n\}$ would not be a null set. But for $t \varepsilon S$, (36) cannot hold. This is a contradiction. Similarly, we prove that $\operatorname*{ess\,inf}_{t \varepsilon (0,2\pi)} q_M(t)$ is bounded too.

Lastly, set

$$\mu_M = \operatorname*{ess\,sup}_{t \varepsilon (0,2\pi)} q_M(t),$$

we shall prove μ_M is bounded in M. Let $\sum = \{s \varepsilon (0,2\pi) | q_M(s) > \frac{\mu_M}{2}\}$. . Then $\operatorname{mes}(\sum) < \frac{2C_8}{\mu_M}$, i.e. $\operatorname{mes}(C\sum) > 2\pi - \frac{2C_8}{\mu_M}$. Let

$t \varepsilon T = \{t \varepsilon (0,2\pi) | q_M(t) > \mu_M - 1\}$. This is not a null set. Substituting into (35), we get

$$\left(2\pi - \frac{2C_8}{\mu_M}\right)\tilde{g}_M\left(-C_5 - 1 + \frac{\mu_M}{2}\right) < 2\pi\, \tilde{g}_M(C_5 + 1).$$

This proves μ_M is bounded. Similarly, we estimate $\operatorname*{ess\,inf}_{t \varepsilon (0,2\pi)} q_M(t)$ i.e. $\|q_M\|_{L^\infty} < C_9$, it implies (31) provided by (32).

The proof is complete.

<u>Remark 1</u> The above method can be extended to attack the following more general problem:

$$\begin{cases} u_{tt} - u_{xx} + g(t,x;u) = 0 \quad \text{for } (x,t) \in (0,\pi) \times (0,T) \\ u(0,t) = u(\pi,t) = 0 \\ u(x,0) = u(x,T) \end{cases}$$

where $T = \frac{2\pi}{\lambda}$, λ is a rational number, and $g(t,x,u)$ is a Baire measurable function defined on $(0,\pi) \times (0,T) \times R^1$, satisfying the following conditions

$(G_1)'$ $g(t,x,0) = 0$, and for fixed (t,x), $g(t,x,\cdot)$ is strictly increasing in u.

$(G_2)'$ $\exists\ \theta \in [0, 1/2)$ and a constant $C_0 > 0$ such that

$$G(t,x,u) = \int_0^u g(t,x,s)ds \le \begin{cases} \theta\, u\, g(t,x,u-0) & \text{for } u > C_0 \\ \\ \theta\, u\, g(t,x,u+0) & \text{for } u < -C_0 . \end{cases}$$

$(G_3)'$ $\lim\limits_{u \to 0} \frac{g(t,x,u)}{u} = 0$ uniformly with respect to

$(t,x) \in (0,\pi) \times (0,T)$.

$(G_4)'$ $g(t,x,u)$ is optimal in the sense discussed in [3] .

Remark 2. We would like to emphasize a difference between [2] and our work. In [2], under slightly weaker growth conditions than (G_2) and $G_3)$, Brezis, Coron and Nirenberg have proved the existence of a nontrivial solution for periods which are small rational multiples of 2π. If $g(t,x,u)$ does not depend on t, of course such a solution is also a 2π periodic solution. But, if $g(t,x,u)$ does depend on T, their method does not seem to work.

We wish to express our grateful thanks to Professor P. Rabinowitz for his kindly help.

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KCC:SL:GCD/db

(14) MRC-TSR-2158

SECURITY CLASSIFICATION OF THIS PAGE (When Data Entered)

REPORT DOCUMENTATION PAGE		READ INSTRUCTIONS BEFORE COMPLETING FORM
1. REPORT NUMBER 2158	2. GOVT ACCESSION NO. AD-A096670	3. RECIPIENT'S CATALOG NUMBER
4. TITLE (and Subtitle) (6) A NEW PROOF AND AN EXTENSION OF A THEOREM OF P. RABINOWITZ CONCERNING NONLINEAR WAVE EQUATIONS.		5. TYPE OF REPORT & PERIOD COVERED Summary Report - no specific reporting period
		6. PERFORMING ORG. REPORT NUMBER
7. AUTHOR(s) (10) K. C. Chang, Shujie Li ~~and~~ G. C. Dong		8. CONTRACT OR GRANT NUMBER(s) (15) DAAG29-80-C-0041
9. PERFORMING ORGANIZATION NAME AND ADDRESS Mathematics Research Center, University of 610 Walnut Street Wisconsin Madison, Wisconsin 53706		10. PROGRAM ELEMENT, PROJECT, TASK AREA & WORK UNIT NUMBERS 1 - Applied Analysis
11. CONTROLLING OFFICE NAME AND ADDRESS U. S. Army Research Office (12) 20 P.O. Box 12211 Research Triangle Park, North Carolina 27709		12. REPORT DATE (11) Dec ~~ember 1980~~ 80
		13. NUMBER OF PAGES 16
14. MONITORING AGENCY NAME & ADDRESS (if different from Controlling Office) (9) Technical summary rept.		15. SECURITY CLASS. (of this report) UNCLASSIFIED
		15a. DECLASSIFICATION/DOWNGRADING SCHEDULE

16. DISTRIBUTION STATEMENT (of this Report)

Approved for public release; distribution unlimited.

17. DISTRIBUTION STATEMENT (of the abstract entered in Block 20, if different from Report)

18. SUPPLEMENTARY NOTES

19. KEY WORDS (Continue on reverse side if necessary and identify by block number)

nonlinear wave equation, mountain pass lemma, critical point, duality argument, generalized gradient

20. ABSTRACT (Continue on reverse side if necessary and identify by block number)

In 1978, P. Rabinowitz proved a theorem concerned with the existence of a non-trivial periodic solution of a nonlinear wave equation with a continuous increasing superlinear nonlinear term. In this paper we present a new and simpler proof of this theorem and relax an assumption on the nonlinear term, which is a discontinuous nondecreasing function.

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